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Lagrangian and Hamiltonian constraints for second-order singular Lagrangians

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Abstract. We study the different kinds of constraints which appear when one deals with singular Lagrangians depending on second-order derivatives. We characterise Ker FL_* and deduce the generalised Hamilton-Dirac equations of motion. The operators relating the Hamiltonian and the Lagrangian constraints are displayed. We extend our results to higher-order singular Lagrangians.

1. Introduction

Recently [1-7] the relation between the Lagrangian and the Hamiltonian formalisms for singular Lagrangians depending on the generalised coordinates and their first-order derivatives has been studied. Although such Lagrangians cover many of the interesting physical theories and models (Yang-Mills theories, relativistic particles and string theories), they do not apply to some important theories such as Hilbert's action for gravity [8] and Podolsky's generalised electrodynamics [9], whose Lagrangians depend on the second-order derivatives of the fields. Also, Polyakov [10] proposed an action for the string theory which has a term proportional to the extrinsic curvature of the world sheet. This idea has been extended to the relativistic particle [11]§.

Some authors [9, 13] have proposed a generalised Dirac-Bergmann algorithm for the second-order case and, recently [14], the study of the Lagrangian and Hamiltonian contraints has received some attention. Nevertheless, the rich constraint structures of these systems have not been analysed and some important points have not been treated in detail.

In this paper, we clarify the relations between the constraints and deduce the Hamiltonian equations of motion, generalising the methods developed in our previous works [1, 2, 4]. In § 2 we present the Euler-Lagrange equations and the generalised Legendre transformation FL. The various spaces involved are shown and Ker FL_* is evaluated, which leads to the determination of the primary Hamiltonian constraints. In § 3, we establish the equations of motion in phase space and in § 4 we obtain the relations between the Hamiltonian and the Lagrangian constraints. In § 5, we illustrate the formalism using a relativistic particle whose action is proportional to its extrinsic curvature. Finally, in § 6 we generalise our results to higher-order singular Lagrangians.

§ After completion of this work, we received [12], where the Hamiltonian analysis of the rigid string proposed in [10] is carried out.

2. The primary Hamiltonian constraints

Let us consider a system described by coordinates $x^A(t)$, $A=1,\ldots,N$, where t is the evolution parameter, and with dynamics given by a Lagrangian $L(x,\dot{x},\ddot{x})$ depending on second-order derivatives. The Euler–Lagrange equations are

$$\frac{\partial L}{\partial x^A}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^A}+\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial \ddot{x}^A}=0 \qquad A=1,\ldots,N \tag{1}$$

which are N fourth-order differential equations that can be written

$$W_{AB}\ddddot{x}^B=\alpha_A \tag{2}$$

where

$$W_{AB}(x,\dot{x},\ddot{x})\equiv\frac{\partial^2 L}{\partial \ddot{x}^A\partial \ddot{x}^B} \tag{3a}$$

is the generalised Hessian matrix and

$$\alpha_A=\alpha_A(x,\dot{x},\ddot{x},\dddot{x}). \tag{3b}$$

If

$$\det W=0 \tag{4}$$

the system is said to be singular and equations (2) cannot be put in normal form. This implies that, in general, the solutions will not be unique for a given set of initial data in T^3Q, where T^3Q is the third-order tangent bundle [15] locally coordinated by x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$, and they will exist only in a submanifold of T^3Q. From now on we assume that

$$\operatorname{rank} W=r<N \qquad N-r\equiv m_1>0 \tag{5}$$

so m_1 nullvectors $\gamma_\mu^A(x,\dot{x},\ddot{x})$ exist such that

$$W_{AB}\gamma_\mu^B=0 \qquad \mu=1,\ldots,m_1. \tag{6}$$

Some Lagrangian constraints follow immediately from (2) and (6)

$$\chi_\mu^{(0)}\equiv\alpha_A\gamma_\mu^A=0 \qquad \chi_\mu^{(0)}\in\Lambda^0(T^3Q). \tag{7}$$

Next we want to find out the consequences of (5) in the phase space. To this end we introduce the generalised Legendre transformation [15]

$$p_{1A}=\frac{\partial L}{\partial \dot{x}^A}-\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \ddot{x}^A} \tag{8a}$$

and

$$p_{2A}=\frac{\partial L}{\partial \ddot{x}^A}. \tag{8b}$$

It is worth noticing that

$$\frac{\partial p_{2A}}{\partial \ddot{x}^B}=W_{AB} \tag{9a}$$

$$\frac{\partial p_{1A}}{\partial \dddot{x}^B}=-W_{AB}. \tag{9b}$$

Roughly speaking, what we want to do is to substitute $\dddot{x}$, $\ddot{x}$ by p_2, p_1. Furthermore, this substitution will be performed by steps and we introduce the spaces

$$\begin{array}{ccccc} & & FL & & \\ T^3Q & \xrightarrow{\alpha_1} & P_1 & \xrightarrow{\alpha_2} & T^*(TQ) \\ (x, \dot{x}, \ddot{x}, \dddot{x}) & & (x, \dot{x}, \ddot{x}, p_1) & & (x, \dot{x}, p_2, p_1) \end{array}$$

where the final space $T^*(TQ)$ has canonical pairs (x, p_1), $(\dot{x}, p_2)$. We will also consider the pull-back

$$FL^* : \Lambda^0(T^*(TQ)) \to \Lambda^0(T^3Q)$$

which amounts to the substitution of p_1, p_2 by x, $\dot{x}$, $\ddot{x}$, $\dddot{x}$ according to equations (8). Due to (6) and (9b), some primary constraints appear in P_1:

$$-\chi^{(0)}_{1\mu} \equiv \gamma^A_\mu \left(p_{1A} - \frac{\partial L}{\partial \dot{x}^A} + \frac{\partial^2 L}{\partial \ddot{x}^A \partial x^B} \dot{x}^B + \frac{\partial^2 L}{\partial \ddot{x}^A \partial \dot{x}^B} \ddot{x}^B \right) \qquad \chi^{(0)}_{1\mu} \in \Lambda^0(P_1) \tag{10}$$

and some others in $T^*(TQ)$, due to (6) and (9a)

$$\phi^{(0)}_\mu \in \Lambda^0(T^*(TQ)) \tag{11}$$

which do not depend on p_1, $\phi^{(0)}_\mu = \phi^{(0)}_\mu(x, \dot{x}, p_2)$. The following identities hold:

$$FL^* \phi^{(0)}_\mu = 0. \tag{12}$$

Taking this into account, we can easily prove that a basis of nullvectors of W is given by

$$\gamma^A_\mu = FL^* \frac{\partial \phi^{(0)}_\mu}{\partial p_{2A}}. \tag{13}$$

Furthermore, we get the relations

$$FL^* \frac{\partial \phi^{(0)}_\mu}{\partial x^A} + \gamma^B_\mu \frac{\partial^2 L}{\partial \ddot{x}^B \partial x^A} = 0 \tag{14a}$$

$$FL^* \frac{\partial \phi^{(0)}_\mu}{\partial \dot{x}^A} + \gamma^B_\mu \frac{\partial^2 L}{\partial \ddot{x}^B \partial \dot{x}^A} = 0. \tag{14b}$$

Next, we shall compute Ker FL_*. If we want that a vector field

$$\Gamma = \alpha^A \frac{\partial}{\partial x^A} + \beta^A \frac{\partial}{\partial \dot{x}^A} + \delta^A \frac{\partial}{\partial \ddot{x}^A} + \varepsilon^A \frac{\partial}{\partial \dddot{x}^A} \tag{15}$$

belongs to Ker FL_*, then it has to obey

$$\Gamma(FL^* x^A) = \Gamma(FL^* \dot{x}^A) = \Gamma(FL^* p_{1A}) = \Gamma(FL^* p_{2A}) = 0 \tag{16}$$

and it follows immediately that $\alpha^A = \beta^A = 0$. Due to (9b) and the fact that p_2 does not depend on $\dddot{x}$, we can take

$$\Gamma_\mu = \gamma^A_\mu \frac{\partial}{\partial \dddot{x}^A} \qquad \mu = 1, \ldots, m_1 \tag{17}$$

but another solution involving $\partial/\partial\ddot{x}$ as well as $\partial/\partial\dddot{x}$ can be chosen:

$$\tilde{\Gamma} = \delta^A \frac{\partial}{\partial \ddot{x}^A} + \varepsilon^A \frac{\partial}{\partial \dddot{x}^A}. \tag{18}$$

Indeed, imposing $\tilde{\Gamma}(FL^* p_{2A}) = 0$, it follows that $\delta^A = \gamma^A_\mu$ and then $\tilde{\Gamma}(FL^* p_{1A}) = 0$ gives

$$\gamma^B_\mu \frac{\partial p_{1A}}{\partial \ddot{x}^B} + \varepsilon^B \frac{\partial p_{1A}}{\partial \dddot{x}^B} = 0 \tag{19}$$

and thus

$$W_{AB}\varepsilon^B = \gamma^B_\mu \frac{\partial p_{1A}}{\partial \ddot{x}^B}. \tag{20}$$

If this system is to have a solution we must impose

$$\gamma^B_\mu \frac{\partial p_{1A}}{\partial \ddot{x}^B} \gamma^A_\nu = 0 \qquad \nu = 1, \ldots, m_1. \tag{21}$$

It is easy to evaluate

$$\frac{\partial p_{1A}}{\partial \ddot{x}^B} = \frac{\partial^2 L}{\partial \ddot{x}^B\, \partial \dot{x}^A} - \frac{\partial^2 L}{\partial \ddot{x}^A\, \partial \dot{x}^B} - \frac{\mathrm{d}}{\mathrm{d}t} W_{AB} \tag{22}$$

and thus the conditions (21) become

$$\gamma^A_\mu \gamma^B_\nu \left(\frac{\partial^2 L}{\partial \ddot{x}^B\, \partial \dot{x}^A} - \frac{\partial^2 L}{\partial \ddot{x}^A\, \partial \dot{x}^B} \right) = 0 \qquad \nu = 1, \ldots, m_1 \tag{23}$$

due to the fact that $\mathrm{d}(\gamma^A_\mu W_{AB} \gamma^B_\nu)/\mathrm{d}t = 0$. It is a simple computation [13] to show that

$$\gamma^A_\mu \gamma^B_\nu \left(\frac{\partial^2 L}{\partial \ddot{x}^B\, \partial \dot{x}^A} - \frac{\partial^2 L}{\partial \ddot{x}^A\, \partial \dot{x}^B} \right) = FL^*\{\phi^{(0)}_\mu, \phi^{(0)}_\nu\} \tag{24}$$

where { , } denotes the Poisson bracket in $T^*(TQ)$ defined by

$$\{F, G\} = \sum_{i=1,2} \left(\frac{\partial F}{\partial q^A_i} \frac{\partial G}{\partial p_{iA}} - \frac{\partial F}{\partial p_{iA}} \frac{\partial G}{\partial q^A_i} \right) \tag{25}$$

where $q^A_1 = x^A$, $q^A_2 = \dot{x}^A$. According to this Poisson bracket, the constraints $\phi^{(0)}_\mu$ can be split into first- and second-class ones [1, 2]:

$$\phi^{(0)}_\mu \begin{cases} \phi^{(0)}_{\mu_0} & \mu_0 = 1, \ldots, m_2 \\ \phi^{(0)}_{\mu'_0} & \mu'_0 = 1, \ldots, m_1 - m_2 \end{cases} \tag{26}$$

with the properties

$$\begin{aligned} &\det\{\phi^{(0)}_{\mu'_0}, \phi^{(0)}_{\nu'_0}\} \underset{M_0}{\neq} 0 \\ &\{\phi^{(0)}_{\mu_0}, \phi^{(0)}_\nu\} \underset{M_0}{=} 0 \end{aligned} \tag{27}$$

M_0 being the submanifold in $T^*(TQ)$ locally defined by $\phi^{(0)}_\mu = 0$. Putting (23), (24), (26) and (27) together, we realise that, in order to construct $\tilde{\Gamma}$, we must restrict ourselves to the nullvectors associated with primary first-class constraints

$$\tilde{\Gamma}_{\mu_0} = \gamma^A_{\mu_0} \frac{\partial}{\partial \ddot{x}^A} + \varepsilon^A_{\mu_0} \frac{\partial}{\partial \dddot{x}^A} \qquad \mu_0 = 1, \ldots, m_2 \tag{28}$$

$\varepsilon^A_{\mu_0}$ being a solution of $W_{AB}\varepsilon^B_{\mu_0} = \gamma^B_{\mu_0}\partial p_{1A}/\partial\ddot{x}^B$. Therefore, on each point of T^3Q, the dimension of Ker FL_* is $m_1 + m_2$. Thus, $FL(T^3Q)$ is a submanifold of $T^*(TQ)$ locally defined by $m_1 + m_2$ independent functions f such that $FL^*f = 0$. From (12) we get m_1 of such functions. The other m_2 are given by those of the primary constraints $\chi^{(0)}_{1\mu}$ in P_1 which are α_2-projectable onto $T^*(TQ)$. Indeed, FL can be decomposed as $FL = \alpha_2 \circ \alpha_1$, in such a way that

$$\alpha_1^*\chi^{(0)}_{1\mu} = 0 \qquad \alpha_2^*\phi^{(0)}_\mu = 0. \tag{29}$$

It is an easy matter to realise that Ker α_{2*} is expanded by the vector fields $\gamma^A_\mu\partial/\partial\ddot{x}^A$ and then, using the relations (24), one concludes that only the $\chi^{(0)}_{1\mu_0}$, $\mu_0 = 1, \ldots, m_2$ are α_2-projectable.

Summing up, the submanifold of primary Hamiltonian constraints in $T^*(TQ)$ is locally defined by

$$\phi^{(0)}_\mu \qquad \mu = 1, \ldots, m_1 \tag{30}$$

and

$$\phi^{(1)}_{\mu_0} \qquad \alpha_2^*\phi^{(1)}_{\mu_0} = \chi^{(0)}_{1\mu_0} \qquad \mu_0 = 1, \ldots, m_2. \tag{31}$$

We notice that the $\varepsilon^A_{\mu_0}$ in (28) are just given by

$$\varepsilon^A_{\mu_0} = FL^* \frac{\partial\phi^{(1)}_{\mu_0}}{\partial p_{2A}}. \tag{32}$$

Some other constraints in $T^*(TQ)$ might appear by requiring the stability of the primary ones.

3. The generalised Hamilton–Dirac equations

First, we want to find the Hamiltonian equations of motion from the Euler–Lagrangre ones. The energy function $E \in \Lambda^0(T^3Q)$

$$E(x, \dot{x}, \ddot{x}, \dddot{x}) = \dot{x}^A FL^*p_{1A} + \ddot{x}^A FL^*p_{2A} - L(x, \dot{x}, \ddot{x}) \tag{33}$$

is FL-projectable since $\Gamma_\mu E = 0$ and $\tilde{\Gamma}_{\mu_0}E = 0$. Therefore, there exists a function $H_c \in \Lambda^0(T^*(TQ))$ such that

$$FL^*H_c = E. \tag{34}$$

This function may be chosen in the following form

$$H_c(x, \dot{x}, p_1, p_2) = \dot{x}^A p_{1A} + \bar{H}_c(x, \dot{x}, p_2). \tag{35}$$

Let us show this point. Since E is also α_1-projectable, we have the intermediate function $E_1 \in \Lambda^0(P_1)$ such that

$$E_1(x, \dot{x}, \ddot{x}, p_1) = \dot{x}^A p_{1A} + \ddot{x}^A \alpha_2^* p_{2A} - L(x, \dot{x}, \ddot{x}).$$

Then, the α_2-projectability of E_1 guarantees the existence of H_c having the form given in (35) which can be written, with the notation of (25), as

$$H_c(q_1, q_2, p_1, p_2) = q_2^A p_{1A} + \bar{H}_c(q_1, q_2, p_2). \tag{35'}$$

Now, derivation of (34) with respect to x gives the empty relation

$$W_{AB}(\dot{x}^B - q_2^B) = 0 \tag{36}$$

while derivation with respect to $\ddot{x}$ shows up

$$W_{AB}\left(\ddot{x}^B - FL^*\frac{\partial H_c}{\partial p_{2B}}\right) = 0$$

and therefore there exist functions λ^μ, completely determined, such that

$$\ddot{x}^A = FL^*\frac{\partial H_c}{\partial p_{2A}} + \lambda^\mu \gamma_\mu^A \tag{37}$$

with γ_μ^A defined by (13).

Once again, derivation of (34) with respect to q_2 gives

$$FL^*\frac{\partial H_c}{\partial q_2^A} + FL^*\frac{\partial H_c}{\partial p_{1B}}\frac{\partial p_{1B}}{\partial q_2^A} + FL^*\frac{\partial H_c}{\partial p_{2B}}\frac{\partial p_{2B}}{\partial q_2^A}$$

$$= \frac{\partial p_{1B}}{\partial q_2^A} q_2^B + FL^* p_{1A} + \frac{\partial p_{2B}}{\partial q_2^A}\ddot{x}^B - \frac{\partial L}{\partial q_2^A}$$

or

$$\frac{\partial L}{\partial q_2^A} - FL^* p_{1A} = -FL^*\frac{\partial H_c}{\partial q_2^A} + \left(\ddot{x}^B - FL^*\frac{\partial H_c}{\partial p_{2B}}\right)\frac{\partial p_{2B}}{\partial q_2^A}$$

and, using (37),

$$\frac{\partial L}{\partial q_2^A} - FL^* p_{1A} = -FL^*\frac{\partial H_c}{\partial q_2^A} + \lambda^\mu FL^*\frac{\partial \phi_\mu^{(0)}}{\partial p_{2B}}\frac{\partial p_{2B}}{\partial q_2^A}$$

which is

$$\frac{\partial L}{\partial q_2^A} - FL^* p_{1A} = -\left(FL^*\frac{\partial H_c}{\partial q_2^A} + \lambda^\mu FL^*\frac{\partial \phi_\mu^{(0)}}{\partial q_2^A}\right) \tag{38}$$

where use has been made of (12). Finally, derivation of (34) with respect to q_1 gives, in a similar way,

$$\frac{\partial L}{\partial q_1^A} = -\left(FL^*\frac{\partial H_c}{\partial q_1^A} + \lambda^\mu FL^*\frac{\partial \phi_\mu^{(0)}}{\partial q_1^A}\right). \tag{39}$$

We are now ready to transform the Lagrangian equations of motion (1) into the Hamiltonian formalism. First, the trajectories satisfy $\mathrm{d}q_1^A/\mathrm{d}t = q_2^A$, which can be written as

$$\frac{\mathrm{d}q_1^A}{\mathrm{d}t} = \frac{\partial H_c}{\partial p_{1A}} + \lambda^\mu \frac{\partial \phi_\mu^{(0)}}{\partial p_{1A}}. \tag{40}$$

Next, (37) is, on the motion,

$$\frac{\mathrm{d}q_2^A}{\mathrm{d}t} = \frac{\partial H_c}{\partial p_{2A}} + \lambda^\mu \frac{\partial \phi_\mu^{(0)}}{\partial p_{2A}}. \tag{41}$$

The definitions (8) allow us to write the left-hand side of (38) as $\mathrm{d}p_2/\mathrm{d}t$, giving

$$\frac{\mathrm{d}p_{2A}}{\mathrm{d}t} = -\frac{\partial H_c}{\partial q_2^A} - \lambda^\mu \frac{\partial \phi_\mu^{(0)}}{\partial q_2^A}. \tag{42}$$

Finally (8*a*) and the Lagrangian equations of motion transform (39) into

$$\frac{\mathrm{d}p_{1A}}{\mathrm{d}t}=-\frac{\partial H_c}{\partial q_1^A}-\lambda^\mu\frac{\partial\phi_\mu^{(0)}}{\partial q_1^A}. \tag{43}$$

Equations (40)-(43) define the time derivative as

$$\frac{\mathrm{d}}{\mathrm{d}t}=\{\ ,H_c\}+\lambda^\mu\{\ ,\phi_\mu^{(0)}\}. \tag{44}$$

The functions λ^μ depend on q_1, q_2 and $\mathrm{d}q_2/\mathrm{d}t$, so the system (40)-(43) is not in normal form. This is just the typical situation of the Hamiltonian equations of motion of a singular system, as is pointed out in [1, 2]. It is possible to prove, following the lines of [2], the equivalence between the Lagrangian equations of motion (1) and the Hamiltonian ones (40)-(43).

We conclude therefore that the canonical study corresponding to second-order singular Lagrangians shows up the same features that we know to happen in the standard case, the only differences being the special form (35) of the Hamiltonian and the fact that the constraints $\phi_\mu^{(0)}$ only depend on half of the momenta.

Next we proceed to study the stability of the primary Hamiltonian constraints. The stability of the $\phi_\mu^{(0)}$ imposes

$$0=\{\phi_\mu^{(0)},H_c\}+\lambda^\nu\{\phi_\mu^{(0)},\phi_\nu^{(0)}\}$$

and using (27) we get

$$\{\phi_{\mu_0}^{(0)},H_c\}=0. \tag{45}$$

These constraints are just the constraints $\phi_{\mu_0}^{(1)}$ defined in (31). To show this, we need to prove that

$$\alpha_2^*\{\phi_{\mu_0}^{(0)},H_c\}=\chi_{1\mu_0}^{(0)}. \tag{46}$$

In fact

$$\alpha_2^*\{\phi_{\mu_0}^{(0)},H_c\}=\alpha_2^*\left(\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q_1^A}q_2^A+\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q_2^A}\frac{\partial H_c}{\partial p_{2A}}-\gamma_{\mu_0}^A\frac{\partial H_c}{\partial q_2^A}\right)$$

but from (12), which can be written as $\alpha_2^*\phi_\mu^{(0)}=0$, we have

$$\alpha_2^*\frac{\partial\phi_{\mu_0}^{(0)}}{\partial q_1^A}=-\alpha_2^*\left(\frac{\partial\phi_{\mu_0}^{(0)}}{\partial p_{2B}}\right)\frac{\partial}{\partial q_1^A}(\alpha_2^*p_{2B})$$

and therefore

$$\alpha_2^*\{\phi_{\mu_0}^{(0)},H_c\}=\gamma_{\mu_0}^A\left(-q_2^B\frac{\partial\alpha_2^*p_{2A}}{\partial q_1^B}-\alpha_2^*\frac{\partial H_c}{\partial p_{2B}}\frac{\partial\alpha_2^*p_{2A}}{\partial q_2^B}-\alpha_2^*\frac{\partial H_c}{\partial q_2^A}\right).$$

Now we use the identities (37) and (38) to substitute $\alpha_2^*\partial H_c/\partial p_2$ and $\alpha_2^*\partial H_c/\partial q_2$ and finally we arrive at

$$\alpha_2^*\{\phi_{\mu_0}^{(0)},H_c\}=\chi_{1\mu_0}^{(0)}+\lambda^\mu FL^*\{\phi_\mu^{(0)},\phi_{\mu_0}^{(0)}\}.$$

But we know from (27) that $FL^*\{\phi_\mu^{(0)},\phi_{\mu_0}^{(0)}\}=0$, and therefore

$$\alpha_2^*\{\phi_{\mu_0}^{(0)},H_c\}=\chi_{1\mu_0}^{(0)}.$$

Now a question of language can be raised. From the point of view of the canonical theory, the primary constraints are the $\phi_\mu^{(0)}$ and the secondary ones are the $\phi_{\mu_0}^{(1)}$; then there may exist tertiary constraints and so on. But if we consider the Legendre transformation given by (8), both $\phi_\mu^{(0)}$ and $\phi_{\mu_0}^{(1)}$ have to be considered as primary canonical constraints. We note that, although constraints $\phi_{\mu_0}^{(1)}$ appear in the canonical formalism by using the dynamics, only equations (40), (41) and (42) are involved; but these equations are equivalent by (15) to the requirement $q_2 = \mathrm{d}q_1/\mathrm{d}t$ and the definitions (8). Therefore, from the Lagrangian point of view, dynamics is not involved (i.e. we make no use of (1)). Thus, there is no kind of paradox. It simply happens that dynamics in canonical formalism includes relations appearing in the Lagrangian formalism as definitions.

4. Connection with the Lagrangian formalism

By generalising the time evolution operator K defined in [2] we can introduce the operators

$$K_2 : \Lambda^0(T^*(TQ)) \to \Lambda^0(P_1) \qquad K_1 : \Lambda^0(P_1) \to \Lambda^0(T^3(TQ))$$

defined by

$$K_2 = \dot{x}^A \alpha_2^* \circ \frac{\partial}{\partial x^A} + \ddot{x}^A \alpha_2^* \circ \frac{\partial}{\partial \dot{x}^A} + \frac{\partial L}{\partial x^A}\, \alpha_2^* \circ \frac{\partial}{\partial p_{1A}} + \left(\frac{\partial L}{\partial \dot{x}^A} - p_{1A}\right) \alpha_2^* \circ \frac{\partial}{\partial p^{2A}} \tag{47}$$

and

$$K_1 = \dot{x}^A \alpha_1^* \circ \frac{\partial}{\partial x^A} + \ddot{x}^A \alpha_1^* \circ \frac{\partial}{\partial \dot{x}^A} + \dddot{x}^A \alpha_1^* \circ \frac{\partial}{\partial \ddot{x}^A} + \frac{\partial L}{\partial x^A}\, \alpha_1^* \circ \frac{\partial}{\partial p_{1A}}. \tag{48}$$

These operators take a function in the correspondent space and give its time derivative in another space. In particular, when acting on constraints in some space, they give the constraints in the other space. In this direction it is easy to prove that

$$K_2 \phi_\mu^{(0)} = \chi_{1\mu}^{(0)} \tag{49}$$

i.e. the known constraints in P_1, and it is equally straightforward to show that

$$K_1 \chi_{1\mu}^{(0)} = \chi_\mu^{(0)} \tag{50}$$

which are the Lagrangian constraints arising directly from (2), i.e. the first generation of Lagrangian constraints. This analysis can be continued at every level of the algorithm to determine the constraints of the theory. Hence, generalising some results of [2, 4], all the Lagrangian constraints can be obtained from the Hamiltonian ones using $K_1 \circ K_2$.

5. The second-order relativistic particle

Let us consider a relativistic particle whose action is proportional to its extrinsic curvature [11, 14]

$$S = \alpha \int_{\tau_1}^{\tau_2} \mathrm{d}\tau (\dot{x}^2)^{-1} [\dot{x}^2 \ddot{x}^2 - (\dot{x}\ddot{x})^2]^{1/2} \equiv \alpha \int_{\tau_1}^{\tau_2} \mathrm{d}\tau (\dot{x}^2)^{-1} \sqrt{g} \tag{51}$$

where an explicit parametrisation has been chosen. The momenta are

$$p_{2\mu}=\frac{\partial L}{\partial \ddot{x}^\mu}=\frac{\alpha}{\dot{x}^2}\frac{1}{\sqrt{g}}[\dot{x}^2\ddot{x}_\mu-(\dot{x}\ddot{x})\dot{x}_\mu]\equiv\frac{\alpha}{\dot{x}^2\sqrt{g}}l_\mu \tag{52}$$

and

$$p_{1\mu}=\frac{\partial L}{\partial \dot{x}^\mu}-\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial \ddot{x}^\mu}=-2\alpha\frac{\sqrt{g}}{(\dot{x}^2)^2}\dot{x}_\mu+\frac{\alpha}{\dot{x}^2\sqrt{g}}[\dot{x}_\mu\ddot{x}^2-(\dot{x}\ddot{x})\ddot{x}_\mu]$$
$$-\frac{\alpha}{\dot{x}^2\sqrt{g}}\dot{l}_\mu+\frac{2\alpha}{(\dot{x}^2)^2\sqrt{g}}l_\mu(\dot{x}\ddot{x})+\frac{3}{2}\frac{\alpha}{\dot{x}^2g^{3/2}}\dot{g} \tag{53}$$

where

$$l^2=g\dot{x}^2 \qquad \dot{l}\dot{x}=-q \qquad l\dot{x}=0. \tag{54}$$

The canonical Hamiltonian (35) is

$$H_c=(p_1q_2). \tag{55}$$

The definition of $p_{2\mu}$ gives the constraints

$$\phi_1^{(0)}=(p_2q_2)=0 \tag{56}$$

$$\phi_2^{(0)}\equiv p_2^2-\alpha^2/q_2^2=0 \tag{57}$$

which are first class. The definition of $p_{1\mu}$ also produces two constraints in P_1, which, due to the fact that both (56) and (57) are first class, are projectable onto $T^*(TQ)$, giving the result

$$\phi_1^{(1)}\equiv(p_1q_2)=0 \tag{58}$$

$$\phi_2^{(1)}\equiv(p_1p_2)=0. \tag{59}$$

One can check that (58) and (59) can be obtained from (56) and (57) using the canonical Hamiltonian (55). The four constraints (56)-(59) constitute the whole set of primary Hamiltonian constraints, and they are all first class. Now we can show that the stability of (58) gives nothing new

$$\dot{\phi}_1^{(1)}=\{\phi_1^{(1)},H_c\}\equiv 0 \tag{60}$$

while (59) brings in a new constraint

$$\dot{\phi}_2^{(1)}=\{\phi_2^{(1)},H_c\}=-p_1^2\equiv-\phi^{(2)} \tag{61}$$

and the algorithm stops here, so finally we are left with five first-class constraints. $\phi^{(2)}$ is a tertiary constraint from the point of view of the Hamiltonian dynamics, but it is a secondary one from the point of view of the Legendre transformation. Its appearance implies that a Lagrangian constraint has to exist, and it can be obtained applying FL^* to $\phi^{(2)}$ or $K_1\circ K_2$ to $\phi_2^{(0)}$.

6. The higher-order case

Here we give a brief summary of the generalisation of the present work to higher-order singular Lagrangians of the type

$$L(x,x^{(1)},x^{(2)},\ldots,x^{(m)}) \qquad x^{(i)}\equiv \mathrm{d}^ix/\mathrm{d}t^i. \tag{62}$$

In phase space we will use the notation $q_i\equiv x^{(i-1)}$.

(a) The Hamiltonian equations of motion (which in the regular case are known as Ostrogradski's equations) are given by [16]

$$\mathrm{d}f/\mathrm{d}t = \{f, H_c\} + \lambda^{\mu}\{f, \phi_{\mu}^{(0)}\} \tag{63}$$

where the canonical Hamiltonian has the form

$$H_c(q_1, \ldots, q_m, p_1, \ldots, p_m) = \sum_{i=1}^{m-1} q_{i+1}p_i - \bar{H}_c(q_1, \ldots, q_m, p_m) \tag{64}$$

the λ^{μ} are canonically unknown functions

$$\lambda^{\mu} = \lambda^{\mu}(q_1, \ldots, q_m, \mathrm{d}q_m/\mathrm{d}t) \tag{65}$$

and the $\phi_{\mu}^{(0)}(q_1, \ldots, q_m, p_m)$ are the canonical primary constraints coming from the definition of the highest-order momenta [15]:

$$p_m = \frac{\partial L}{\partial x^{(m)}}. \tag{66}$$

The other momenta are defined according to

$$p_i = \frac{\partial L}{\partial x^{(i)}} - \frac{\mathrm{d}}{\mathrm{d}t} p_{i+1} \qquad i = 1, \ldots, m-1. \tag{67}$$

(b) The Legendre transformation given by the definition of the whole set of momenta leads to a family of constraints (including the $\phi_{\mu}^{(0)}$) which splits in m generations. All these constraints can be obtained canonically by requiring the stability of the previous generation. This process may determine some of the functions λ^{μ}. After the last generation of primary Hamiltonian constraints has been obtained, new constraints may appear which are not a consequence of the Legendre transformation.

(c) A number of intermediate spaces, $P_1, P_2, \ldots, P_{m-1}$, can be defined to pass from $T^{2m-1}Q$ to $T^*(T^{m-1}Q)$:

$$\begin{aligned}
&T^{2m-1}Q \to P_1 \ldots P_{m-1} \to T^*(T^{m-1}Q)\\
&(x^{(0)} \ldots x^{(2m-1)}) \quad (x^{(0)} \ldots x^{(2m-2)}p_1) \ldots (x^{(0)} \ldots x^{(m)}p_1 \ldots p_{m-1})\\
&\qquad (x^{(0)} \ldots x^{(m-1)}p_1 \ldots p_m).
\end{aligned} \tag{68}$$

Some constraints appear in these spaces due to the singular character of the Lagrangian.

(d) A family of operators K_i, which connect the constraints in the different spaces mentioned above, can be defined in the same way as in the second-order case.

7. Conclusions

In this paper we have generalised the results of [1, 2] to the case of singular Lagrangians depending on second-order derivatives. An important result is that the primary Hamiltonian constraints, i.e. those which follow directly from the definition of the Legendre transformation, come in two generations. The first one is brought in by the definition of the p_{2A} and the second one comes from the definition of some of the p_{1A}. Hamiltonian dynamics gives the second generation from the first one. Introducing the operators K_1 and K_2 we have been able to relate the constraints appearing in T^3Q, P_1 and $T^*(TQ)$. All the results have a straightforward generalisation to higher-order singular Lagrangians.

We expect this work will be useful in order to construct the canonical gauge transformations and the BRST generators for higher-order systems.

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References

[1] Batlle C, Gomis J, Pons J M and Román-Roy N 1987 *Lett. Math. Phys.* **13** 17

[2] Batlle C, Gomis J, Pons J M and Román-Roy N 1986 *J. Math. Phys.* **27** 2953

[3] Batlle C, Gomis J, Pons J M and Román-Roy N 1987 *J. Phys. A: Math. Gen.* **20** 5113

[4] Pons J M 1988 *J. Phys. A: Math. Gen.* **21** 2705

[5] Nesterenko V V and Chervyakov A M 1986 *Theor. Math. Phys.* **64** 701

[6] Cariñena J F, López C and Román-Roy N 1987 *Origin of the Lagrangian constraints and their relation with the Hamiltonian formulation* and *Geometric study on the connection between the Lagrangian and the Hamiltonian constraints, Preprints* University of Zaragoza

[7] Gotay M J and Nester J M 1979 *Ann. Inst. H Poincaré* A **30** 129; 1980 *Ann. Inst. H Poincaré* A **32** 1

[8] Dresden M and Dutt S K 1986 *Pure gravity as a constrained second-order system. ITP preprint*

[9] Galvao C and Pimentel A 1986 *The canonical structure of Podolsky generalised electrodynamics. IAS preprint*

[10] Polyakov A M 1986 *Nucl. Phys.* B **268** 406

[11] Pisarski R D 1986 *Phys. Rev.* D **34** 670

[12] Nesterenko V V and Nguyen Suan Han 1987 *Dubna preprint* P2-87-363

[13] Tapia V 1985 *Nuovo Cimento* B **90** **15**

[14] Nesterenko V V 1987 *Singular Lagrangians with higher-order derivatives. Dubna preprint* E2-87-9

[15] de León M and Rodrigues P R 1985 *Generalized Classical Mechanics and Field Theory* (*North-Holland Mathematics Studies* **112**) (Amsterdam: Elsevier)

[16] Pons J M 1988 *Ostrogradski's theorem for higher order singular Lagrangians. Barcelona preprint* UBECM-PF-5/88